

THE QUANTUM ORBIFOLD COHOMOLOGY OF TORIC STACK BUNDLES

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ABSTRACT. We study Givental's Lagrangian cone for the quantum orbifold cohomology of toric stack bundles. Using Gromov-Witten invariants of the base and combinatorics of the toric stack fibers, we construct an explicit slice of the Lagrangian cone defined by the genus 0 Gromov-Witten theory of a toric stack bundle.

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1. INTRODUCTION

An important problem in Gromov-Witten theory is the computations of Gromov-Witten invariants of orbifolds. Genus 0 Gromov-Witten invariants of toric stacks can be determined via a Givental-style mirror theorem proven in [8] and [7], while their higher genus Gromov-Witten invariants can be determined by Givental-Teleman reconstruction of semi-simple CohFTs [10], [17]. Toric bundles over a base B are studied in [16], where their cohomology rings were computed. Assuming knowledge about Gromov-Witten invariants of B , genus 0 Gromov-Witten invariants of a toric bundle over B can be determined via the mirror theorem in [5], while their higher genus Gromov-Witten invariants can be determined from genus 0 invariants and localization [6].

Toric stack bundles, introduced by Jiang [13], generalize toric bundles by using toric Deligne-Mumford stacks as fibers. The main result of this paper, Theorem 3.5, is a mirror theorem for toric stack bundles $\mathcal{P} \rightarrow B$. Roughly speaking, Theorem 3.5 gives an explicit slice, the extended I -function, of the Lagrangian cone $\mathcal{L}_{\mathcal{P}}$ of the genus 0 Gromov-Witten theory of \mathcal{P} , which can be used to determine all genus 0 Gromov-Witten invariants of \mathcal{P} following [11], assuming that genus 0 Gromov-Witten invariants of B are known.

Theorem 3.5 generalizes the mirror theorems in [5], [8]. Our proof of Theorem 3.5 follows the same approach as those in [5], [8]: localization yields a characterization result of the Lagrangian cone $\mathcal{L}_{\mathcal{P}}$, see Theorem 4.1. We prove that the extended I -function lies on $\mathcal{L}_{\mathcal{P}}$ by checking the conditions **(C1)**–**(C3)** in Theorem 4.1. The verification of **(C3)** for toric stack bundles involves a novel point. **(C3)** concerns fixed points of the fiberwise torus action on \mathcal{P} . Components of the fixed loci are abelian gerbes over the base B . To check **(C3)**, we need to know Gromov-Witten theory of certain abelian gerbes over B . Fortunately these were previously solved in [3].

The result in this paper will have applications to study birational transformation of orbifold Gromov-Witten invariants. An important class of crepant birational transformation of varieties is *flops*. In the study of ordinary flops as in [14], the local models, which are toric bundles over a base scheme B with fibre the projective bundle over a projective space, played an important role in the proof of invariance of genus zero Gromov-Witten invariants. A special example in our case is a toric stack bundle with fibre the weighted projective bundle over a weighted projective stack, which is the local model of *ordinary orbifold flop*. Theorem 3.5 will play a crucial role to prove the crepant transformation conjecture for ordinary orbifold flops.

The rest of the paper is organized as follows. Section 2.1 contains a brief review of genus 0 Gromov-Witten theory. Section 2.2 contains a review about toric stacks and related materials. The construction of toric stack bundles is recalled in Section 3. In Section 4 we apply localization to derive a characterization result of the Lagrangian cone for toric stack bundles. The main result is then proven in Section 5.

Throughout this paper, we work over \mathbb{C} .

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2. PREPARATORY MATERIALS

2.1. Gromov-Witten theory. We give a very brief account on Gromov-Witten theory. The materials we need are discussed in more details in [8, Section 2], to which we refer the reader.

Let \mathcal{X} be a smooth proper Deligne-Mumford stack with projective¹ coarse moduli space X . The *Chen-Ruan orbifold cohomology* $H_{\text{CR}}^*(\mathcal{X})$ of \mathcal{X} is additively the cohomology of the inertia stack $\mathcal{IX} := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$, where the fiber product is taken over the diagonal. The grading of $H_{\text{CR}}^*(\mathcal{X})$ is the usual grading on cohomology shifted by *age*. $H_{\text{CR}}^*(\mathcal{X})$ is also equipped with a non-degenerate pairing $(-, -)_{\text{CR}}$ called orbifold Poincaré pairing.

Gromov-Witten invariants of \mathcal{X} are defined as the following intersection numbers:

$$\langle a_1 \bar{\psi}_1^{k_1}, \dots, a_n \bar{\psi}_n^{k_n} \rangle_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)]^w} (\text{ev}_1^* a_1) \bar{\psi}_1^{k_1}, \dots, \text{ev}_n^* (a_n) \bar{\psi}_n^{k_n},$$

where

- $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ is the moduli stack of n -pointed genus g degree d stable maps to \mathcal{X} with sections to all marked gerbes.
- $[\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)]^w \in H_*(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d), \mathbb{Q})$ is the weighted virtual fundamental class, which is a multiple of the usual virtual fundamental class. More details can be found in [1, Section 4.6] and [18, Section 2.5.1].
- For $i = 1, \dots, n$, $\text{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathcal{X}, d) \rightarrow \mathcal{IX}$ is the evaluation map.
- For $i = 1, \dots, n$, $\bar{\psi}_i \in H^2(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d), \mathbb{Q})$ are the descendant classes.
- $a_1, \dots, a_n \in H^*(\mathcal{IX})$.

Gromov-Witten invariants can be packaged into generating functions, as follows. The genus g Gromov-Witten potential of \mathcal{X} is

$$\mathcal{F}_{\mathcal{X}}^g(\mathbf{t}) := \sum_{n,d} \frac{Q^d}{n!} \langle \mathbf{t}, \dots, \mathbf{t} \rangle_{g,n,d},$$

where Q^d is an element in the Novikov ring of \mathcal{X} , $\mathbf{t} = \mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \dots \in H_{\text{CR}}^*(\mathcal{X})[z]$, and $\langle \mathbf{t}, \dots, \mathbf{t} \rangle_{g,n,d} := \sum_{k_1, \dots, k_n} \langle t_{k_1} \bar{\psi}^{k_1}, \dots, t_{k_n} \bar{\psi}^{k_n} \rangle_{g,n,d}$.

We briefly recall the Givental's formalism about the orbifold Gromov-Witten invariants in terms of a Lagrangian cone in certain symplectic vector space, which was developed in [18]. Let

$$\mathcal{H} := H_{\text{CR}}^*(\mathcal{X}, \mathbb{C}) \otimes \mathbb{C}[[\overline{\text{NE}}(\mathcal{X})]][[z, z^{-1}]],$$

where $\overline{\text{NE}}(\mathcal{X})$ is the Mori cone of \mathcal{X} . There is a $\mathbb{C}[[\overline{\text{NE}}(\mathcal{X})]]$ -valued symplectic form

$$\Omega(f, g) := \text{Res}_{z=0}(f(-z), g(z))_{\text{CR}} dz,$$

where $(-, -)_{\text{CR}}$ is the orbifold Poincaré pairing. Let

$$\mathcal{H}_+ = H_{\text{CR}}^*(\mathcal{X}, \mathbb{C}) \otimes \mathbb{C}[[\overline{\text{NE}}(\mathcal{X})]][[z]] \text{ and } \mathcal{H}_- = z^{-1} H_{\text{CR}}^*(\mathcal{X}, \mathbb{C}) \otimes \mathbb{C}[[\overline{\text{NE}}(\mathcal{X})]][[z^{-1}]].$$

¹In the presence of a torus action, we may allow \mathcal{X} to be only semi-projective.

Then $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and one can think of $\mathcal{H} = T^*(\mathcal{H}_+)$.

The graph of the differential of $\mathcal{F}_{\mathcal{X}}^0$, in the dilaton-shifted coordinates, defined a Lagrangian submanifold $\mathcal{L}_{\mathcal{X}}$ inside the symplectic vector space \mathcal{H} , more explicitly,

$$\mathcal{L}_{\mathcal{X}} := \{(p, q) \in \mathcal{H}_- \oplus \mathcal{H}_+ | p = d_q \mathcal{F}_{\mathcal{X}}^0\} \subset \mathcal{H}.$$

Tautological equations for genus 0 Gromov-Witten invariants imply that $\mathcal{L}_{\mathcal{X}}$ is a cone ruled by a *finite dimensional* family of affine subspaces. A particularly important finite-dimensional slice of $\mathcal{L}_{\mathcal{X}}$ is the *J-function*:

$$J_{\mathcal{X}}(t, z) := 1z + t + \sum_{n,d} \sum_{\alpha} \frac{Q^d}{n!} \langle t, \dots, t, \frac{\phi_{\alpha}}{z - \psi} \rangle_{0,n+1,d} \phi^{\alpha},$$

where $t = \sum_{\alpha} t^{\alpha} \phi_{\alpha} \in H_{\text{CR}}^*(\mathcal{X})$ and $\{\phi_{\alpha}\}, \{\phi^{\alpha}\} \subset H_{\text{CR}}^*(\mathcal{X})$ are additive bases dual to each other under $(-, -)_{\text{CR}}$.

Another important slice of $\mathcal{L}_{\mathcal{X}}$ is given by the *S-operator*:

$$(1) \quad S_{\mathcal{X}}(t, z)(\gamma) := \gamma + \sum_{n,d} \sum_{\alpha} \frac{Q^d}{n!} \langle t, \dots, t, \gamma, \frac{\phi_{\alpha}}{z - \psi} \rangle_{0,n+2,d} \phi^{\alpha},$$

where $\gamma \in H_{\text{CR}}^*(\mathcal{X}; \mathbb{Q})$.

The discussion here extends with little efforts to equivariant and twisted settings.

2.2. Preliminaries on toric stacks. In this section we collect some basic materials concerning toric stacks. Our presentation closely follows [8, Section 3].

2.2.1. Construction. A toric Deligne-Mumford stack is defined by a stacky fan $\Sigma = (\mathbf{N}, \Sigma, \rho)$, where

- \mathbf{N} is a finitely generated abelian group of rank r ;
- $\Sigma \subset \mathbf{N}_{\mathbb{Q}} = \mathbf{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a rational simplicial fan;
- $\rho : \mathbb{Z}^n \rightarrow \mathbf{N}$ is a map given by $\{\rho_1, \dots, \rho_n\} \subset \mathbf{N}$, which is assumed to have finite cokernel. The ρ_i 's are vectors determining the rays of the stacky fan.

Let $\bar{\rho}_i$ be the image of ρ_i under the natural map $\mathbf{N} \rightarrow \mathbf{N}_{\mathbb{Q}}$.

The *fan sequence* is

$$(2) \quad 0 \longrightarrow \mathbb{L} := \ker(\rho) \longrightarrow \mathbb{Z}^n \xrightarrow{\rho} \mathbf{N}.$$

Let $\rho^{\vee} : (\mathbb{Z}^*)^n \rightarrow \mathbb{L}^{\vee}$ be the Gale dual of ρ , where $\mathbb{L}^{\vee} := H^1(\text{Cone}(\rho)^*)$ is an extension of $\mathbb{L}^* = \text{Hom}(\mathbb{L}, \mathbb{Z})$ by a torsion subgroup. More details can be found in [4]. The *divisor sequence* is

$$(3) \quad 0 \longrightarrow \mathbf{N}^* \xrightarrow{\rho^*} (\mathbb{Z}^*)^n \xrightarrow{\rho^{\vee}} \mathbb{L}^{\vee}.$$

Applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^{\times})$ to the dual map ρ^{\vee} yields a homomorphism

$$\alpha : G \rightarrow (\mathbb{C}^{\times})^n, \quad \text{where } G := \text{Hom}_{\mathbb{Z}}(\mathbb{L}^{\vee}, \mathbb{C}^{\times}),$$

and we let G act on \mathbb{C}^n via this homomorphism.

For $I \subset \{1, 2, \dots, n\}$, let σ_I be the cone generated by $\bar{\rho}_i, i \in I$ and let \bar{I} be the complement of I in $\{1, 2, \dots, n\}$. The collection of *anti-cones* \mathcal{A} is defined as follows:

$$\mathcal{A} := \{I \subset \{1, 2, \dots, n\} : \sigma_{\bar{I}} \in \Sigma\}.$$

For $I \subset \{1, \dots, n\}$, define

$$\mathbb{C}^I = \{(z_1, \dots, z_n) : z_i = 0 \text{ for } i \notin I\}.$$

Let \mathcal{U} be the open subset of \mathbb{C}^n defined as

$$\mathcal{U} := \mathbb{C}^n \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I.$$

Definition 2.1 (see [4], [12]). The toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is defined as the quotient stack

$$\mathcal{X}(\Sigma) := [\mathcal{U}/G].$$

Throughout this paper we assume the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ has semi-projective coarse moduli space, that is, the coarse moduli space $X(\Sigma)$ is a toric variety that has at least one torus-fixed point and the natural map $X(\Sigma) \rightarrow \text{Spec } H^0(X(\Sigma), \mathcal{O}_{X(\Sigma)})$ is projective. See [8, Section 3.1] for more details.

Definition 2.2 ([4]). Given a stacky fan $\Sigma = (\mathbf{N}, \Sigma, \rho)$, we define the set of box elements $\text{Box}(\Sigma)$ as follows

$$\text{Box}(\sigma) := \left\{ b \in \mathbf{N} : \bar{b} = \sum_{\rho_k \subseteq \sigma} c_k \bar{\rho}_k \text{ for some } 0 \leq c_k < 1 \right\}$$

And set $\text{Box}(\Sigma) := \bigcup_{\sigma \in \Sigma} \text{Box}(\sigma)$

The connected components of the inertia stack $\mathcal{IX}(\Sigma)$ are indexed by the elements of $\text{Box}(\Sigma)$ (see [4]). Moreover, given $b \in \text{Box}(\Sigma)$, the age of the corresponding connected component of \mathcal{IX} is defined by $\text{age}(b) := \sum_{\rho_k \subseteq \sigma} c_k$.

The Picard group $\text{Pic}(\mathcal{X}(\Sigma))$ of $\mathcal{X}(\Sigma)$ can be identified with the character group $\text{Hom}(G, \mathbb{C}^\times)$. Hence

$$(4) \quad \mathbb{L}^\vee = \text{Hom}(G, \mathbb{C}^\times) \cong \text{Pic}(\mathcal{X}(\Sigma)) \cong H^2(\mathcal{X}(\Sigma); \mathbb{Z}).$$

The inclusion $(\mathbb{C}^\times)^n \subset \mathcal{U}$ induces an open embedding of the stack $\mathcal{T} = [(\mathbb{C}^\times)^n/G]$ into $\mathcal{X}(\Sigma)$ and we have $\mathcal{T} \cong \mathbb{T} \times B\mathbf{N}_{\text{tor}}$ with $\mathbb{T} := (\mathbb{C}^\times)^n/\text{Im}(\alpha) \cong \mathbf{N} \otimes \mathbb{C}^\times$ and $\mathbf{N}_{\text{tor}} \cong \ker(\alpha)$. The Picard stack \mathcal{T} acts naturally on $\mathcal{X}(\Sigma)$ and restricts to the \mathbb{T} -action on $\mathcal{X}(\Sigma)$. A \mathcal{T} -equivariant line bundle on $\mathcal{X}(\Sigma)$ corresponds to a $(\mathbb{C}^\times)^n$ -equivariant line bundle on \mathcal{U} . Thus,

$$\text{Pic}_{\mathcal{T}}(\mathcal{X}(\Sigma)) \cong \text{Hom}((\mathbb{C}^\times)^n, \mathbb{C}^\times) \cong (\mathbb{Z}^n)^*.$$

We write u_1, \dots, u_n for the basis of \mathcal{T} -equivariant line bundles on $\mathcal{X}(\Sigma)$ corresponding to the standard basis of $(\mathbb{Z}^n)^*$ and write D_1, \dots, D_n for the corresponding non-equivariant line bundles, i.e.

$$D_i = \rho^\vee(u_i).$$

By abuse of notation, we also write u_i and D_i for the corresponding first Chern classes.

2.2.2. *S-extended stacky fan.* Given a stacky fan $\Sigma = (\mathbf{N}, \Sigma, \rho)$ and a finite set

$$S = \{s_1, \dots, s_m\} \subset \mathbf{N}.$$

The S -extended stacky fan in the sense of [13] is given by $(\mathbf{N}, \Sigma, \rho^S)$, where

$$(5) \quad \rho^S : \mathbb{Z}^{n+m} \rightarrow \mathbf{N}, \quad \rho^S(e_i) := \begin{cases} \rho_i & 1 \leq i \leq n; \\ s_{i-n} & n+1 \leq i \leq n+m. \end{cases}$$

Let \mathbb{L}^S be the kernel of $\rho^S : \mathbb{Z}^{n+m} \rightarrow \mathbf{N}$. Gale duality of the S -extended fan sequence

$$(6) \quad 0 \longrightarrow \mathbb{L}^S := \ker(\rho^S) \longrightarrow \mathbb{Z}^{n+m} \xrightarrow{\rho^S} \mathbf{N}$$

yields the S -extended divisor sequence

$$(7) \quad 0 \longrightarrow \mathbf{N}^* \xrightarrow{\rho^*} (\mathbb{Z}^*)^{n+m} \xrightarrow{\rho^{S\vee}} (\mathbb{L}^S)^\vee,$$

where $(\mathbb{L}^S)^\vee$ is the Gale dual of ρ^S . As in [8, Section 4], $(\mathbb{L}^S)^\vee$ is the S -extended Picard group of $\mathcal{X}(\Sigma)$.

Let \mathcal{A}^S be the collection of S -extended anti-cones, i.e.

$$\mathcal{A}^S := \{I^S \subset \{1, 2, \dots, n+m\} : \sigma_{I^S} \in \Sigma\}.$$

Note that

$$\{s_1, \dots, s_m\} \subset I^S, \quad \forall I^S \in \mathcal{A}^S.$$

By applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to the S -extended dual map ρ^\vee , we have a homomorphism

$$\alpha^S : G^S \rightarrow (\mathbb{C}^\times)^{n+m}, \quad \text{where } G^S := \text{Hom}_{\mathbb{Z}}((\mathbb{L}^S)^\vee, \mathbb{C}^\times).$$

Define \mathcal{U}^S to be the open subset of \mathbb{C}^{n+m} defined by \mathcal{A}^S :

$$\mathcal{U}^S := \mathbb{C}^{n+m} \setminus \bigcup_{I^S \notin \mathcal{A}^S} \mathbb{C}^{I^S} = \mathcal{U} \times (\mathbb{C}^\times)^m,$$

where

$$\mathbb{C}^{I^S} = \{(z_1, \dots, z_{n+m}) : z_i = 0 \text{ for } i \notin I^S\}.$$

Let G^S act on \mathcal{U}^S via α^S . Then we obtain the quotient stack $[\mathcal{U}^S/G^S]$. Jiang [13] showed that

$$[\mathcal{U}^S/G^S] \cong [\mathcal{U}/G] = \mathcal{X}(\Sigma).$$

2.2.3. *Toric maps from \mathbb{P}_{r_1, r_2} to $\mathcal{X}(\Sigma)$.* We recall the discussion in [8, Section 3.5] on maps from 1-dimensional toric stacks to a toric stack. For positive integers r_1 and r_2 let \mathbb{P}_{r_1, r_2} be the unique toric Deligne-Mumford stack such that

- its coarse moduli space is \mathbb{P}^1 ;
- its isotropy group at $0 \in \mathbb{P}^1$ is μ_{r_1} ;
- its isotropy group at $\infty \in \mathbb{P}^1$ is μ_{r_2} ; and
- there are no non-trivial orbifold structures at other points.

As in [4], For an extended stacky fan Σ , let $\sigma \in \Sigma$ be a cone, define

$$\text{link}(\sigma) := \{\tau : \sigma + \tau \in \Sigma, \sigma \cap \tau = 0\},$$

and $\rho_{i_1}, \dots, \rho_{i_l}$ be the rays in $\text{link}(\sigma)$. A cone $\sigma \in \Sigma$ defines a closed substack of $\mathcal{X}(\Sigma)$, which is the toric stack $\mathcal{X}(\Sigma/\sigma)$ corresponding to the quotient stacky fan $(\mathbf{N}(\sigma), \Sigma/\sigma, \rho(\sigma))$, where Σ/σ is the quotient fan in $\mathbf{N}(\sigma)_{\mathbb{Q}} = (\mathbf{N}/\sum_{i \in \sigma} \mathbb{Z}\rho_i) \otimes \mathbb{Q}$. More precisely, $\Sigma/\sigma = (\mathbf{N}(\sigma), \Sigma/\sigma, \rho(\sigma))$

is an extended stacky fan, where $\rho(\sigma) : \mathbb{Z}^{l+m} \rightarrow \mathbf{N}(\sigma)$ is given by the images of $\rho_{i_1}, \dots, \rho_{i_l}, s_1, \dots, s_m$ under $\mathbf{N} \rightarrow \mathbf{N}(\sigma)$. From the construction of extended toric Deligne-Mumford stack, we have

$$\mathcal{X}(\Sigma/\sigma) := [\mathcal{U}^S(\sigma)/G^S(\sigma)]$$

where $\mathcal{U}^S(\sigma) = (\mathbb{C}^l - V(J_{\Sigma/\sigma})) \times (\mathbb{C}^\times)^m = \mathcal{U}(\sigma) \times (\mathbb{C}^\times)^m$, $G^S(\sigma) = \text{Hom}_{\mathbb{Z}}(\mathbb{L}^{S^\vee}(\sigma), \mathbb{C}^\times)$.

For a box element $b \in \text{Box}(\Sigma)$, let $\mathcal{X}(\Sigma)_b$ be the component of the inertia stack $\mathcal{IX}(\Sigma)$ corresponding to b . Then $\mathcal{X}(\Sigma)_b \cong \mathcal{X}(\Sigma/\sigma(b))$, where $\sigma(b)$ is the minimal cone containing \bar{b} . We define $b_i \in [0, 1]$, $1 \leq i \leq n$ by the condition $\bar{b} = \sum_{i=1}^n b_i \bar{\rho}_i$, note that $b_i = 0$ for $\bar{\rho}_i \notin \sigma(b)$.

Definition 2.3 (see [8], Notation 8). Let $\sigma, \sigma' \in \Sigma$ be two top dimensional cones, we write $\sigma \dagger \sigma'$ if they intersect along a codimension-1 face and we denote j to be the unique index such that $\bar{\rho}_j \in \sigma \setminus \sigma'$, and j' to be the unique index such that $\bar{\rho}_{j'} \in \sigma' \setminus \sigma$.

Proposition 2.4 ([8], Proposition 10). *Let $\mathcal{X}(\Sigma)$ be the toric Deligne-Mumford stack associated to a stacky fan $\Sigma = (\mathbf{N}, \Sigma, \rho)$. Suppose top dimensional cones σ, σ' satisfy $\sigma \dagger \sigma'$ and $b \in \text{Box}(\sigma)$. The following are equivalent:*

- A positive rational number c such that $\langle c \rangle = \hat{b}_j$, where $\hat{b} = \text{inv}(b)$ is the involution of b .
- A representable toric morphism $f : \mathbb{P}_{r_1, r_2} \rightarrow \mathcal{X}(\Sigma)$ such that $f(0) = \mathcal{X}(\Sigma/\sigma)$, $f(\infty) = \mathcal{X}(\Sigma/\sigma')$ and the restriction $f|_0 : B\mu_{r_1} \rightarrow \mathcal{X}(\Sigma/\sigma)$ gives the box element $\hat{b} \in \text{Box}(\sigma)$.

The data σ, σ', b and c determine the map $f : \mathbb{P}_{r_1, r_2} \rightarrow \mathcal{X}(\Sigma)$ and determine the rational number r_2 and the box element $b' \in \text{Box}(\sigma')$ given by the restriction $f|_\infty : B\mu_{r_2} \rightarrow \mathcal{X}(\Sigma)$. More precisely, b' is the unique element of $\text{Box}(\sigma')$ such that

$$(8) \quad \hat{b} + \lfloor c \rfloor \rho_j + q' \rho_{j'} + b' \equiv 0 \pmod{\bigoplus_{i \in \sigma \cap \sigma'} \mathbb{Z} \rho_i}$$

for some $q' \in \mathbb{Z}_{\geq 0}$. As in [8, Definition 12], define $d_{c, \sigma, j}$ to be the element of $\mathbb{L} \otimes \mathbb{Q}$ satisfying the relation

$$c \bar{\rho}_j + \left(\sum_{i \in \sigma \cap \sigma'} c_i \bar{\rho}_i \right) + c' \bar{\rho}_{j'} = 0$$

such that

$$D_j \cdot d_{c, \sigma, j} = c, \quad D_{j'} \cdot d_{c, \sigma, j} = c', \quad D_i \cdot d_{c, \sigma, j} = c_i \text{ for } i \in \sigma \cap \sigma',$$

and

$$D_i \cdot d_{c, \sigma, j} = 0 \text{ for } i \notin \sigma \cup \sigma'.$$

Hence, $d_{c, \sigma, j}$ is the degree of the representable toric morphism $f : \mathbb{P}_{r_1, r_2} \rightarrow \mathcal{X}(\Sigma)$. Let $\Lambda E_{\sigma, b}^{\sigma', b'} \subset \mathbb{L} \otimes \mathbb{Q}$ to be the set of degrees $d_{c, \sigma, j}$ representable toric morphisms $f : \mathbb{P}_{r_1, r_2} \rightarrow \mathcal{X}(\Sigma)$ such that $f(0) = \mathcal{X}(\Sigma/\sigma)$, $f(\infty) = \mathcal{X}(\Sigma/\sigma')$ and $f|_0$ and $f|_\infty$ give the box elements \hat{b} and b' , respectively. More precisely,

$$\Lambda E_{\sigma, b}^{\sigma', b'} = \left\{ d_{c, \sigma, j} \in \mathbb{L} \otimes \mathbb{Q} : c > 0 \text{ such that } \langle c \rangle = \hat{b}_j \text{ and } b' \text{ satisfies (8)} \right\},$$

see [8, Definition 14].

We recall a few notions related to extended degrees for toric stacks.

Definition 2.5 ([8], Definition 22). Consider a cone $\sigma \in \Sigma$, let $\Lambda_\sigma^S \subset \mathbb{L}^S \otimes \mathbb{Q} \subset \mathbb{Q}^{n+m}$ be the set of elements $\lambda = \sum_{i=1}^{n+m} \lambda_i e_i$ such that

$$\lambda_{n+j} \in \mathbb{Z}, \quad 1 \leq j \leq m; \quad \lambda_i \in \mathbb{Z}, \text{ if } i \notin \sigma \text{ and } 1 \leq i \leq n.$$

Set $\Lambda^S := \cup_{\sigma \in \Sigma} \Lambda_\sigma^S$.

Definition 2.6 ([8], Definition 23). The reduction function v^S is defined by

$$\begin{aligned} v^S : \Lambda^S &\longrightarrow \text{Box}(\Sigma) \\ \lambda &\longmapsto \sum_{i=1}^n [\lambda_i] \rho_i + \sum_{j=1}^m [\lambda_{n+j}] s_j \end{aligned}$$

Hence, we have $\overline{v^S(\lambda)} = \sum_{i=1}^n \langle -\lambda_i \rangle \bar{\rho}_i \in \sigma$ for $\lambda \in \Lambda_\sigma^S$. We introduce the following sets:

$$\begin{aligned} \Lambda_b^S &:= \{\lambda \in \Lambda^S : v^S(\lambda) = b\} \\ \Lambda E^S &:= \Lambda^S \cap \overline{\text{NE}}^S(\mathcal{X}(\Sigma)) \\ \Lambda E_b^S &:= \Lambda_b^S \cap \overline{\text{NE}}^S(\mathcal{X}(\Sigma)) \end{aligned}$$

3. TORIC STACK BUNDLES

3.1. Construction. Let $P \rightarrow B$ be a principal $(\mathbb{C}^\times)^{n+m}$ -bundle over a smooth projective variety B , we introduce the toric stack bundle ${}^P\mathcal{X}(\Sigma)$.

Definition 3.1 ([13]). The toric stack bundle $\pi : \mathcal{P} := {}^P\mathcal{X}(\Sigma) \rightarrow B$ is defined to be the quotient stack

$${}^P\mathcal{X}(\Sigma) := [(P \times_{(\mathbb{C}^\times)^{n+m}} \mathcal{U}^S) / G^S]$$

where G^S acts on P trivially.

It is shown in [13] that \mathcal{P} is a smooth Deligne-Mumford stack.

We now recall the description of the inertia stack of \mathcal{P} . We have an action of $(\mathbb{C}^\times)^{n+m}$ on $\mathcal{U}^S(\sigma)$ induced by the natural action of $(\mathbb{C}^\times)^{l+m}$ on $\mathcal{U}^S(\sigma)$ and the projection $(\mathbb{C}^\times)^{n+m} \rightarrow (\mathbb{C}^\times)^{l+m}$. We let

$$\begin{aligned} {}^P\mathcal{X}(\Sigma/\sigma) &= [(P \times_{(\mathbb{C}^\times)^{n+m}} (\mathbb{C}^\times)^{l+m} \times_{(\mathbb{C}^\times)^{l+m}} \mathcal{U}^S(\sigma)) / G^S(\sigma)] \\ &= [(P \times_{(\mathbb{C}^\times)^{n+m}} \mathcal{U}^S(\sigma)) / G^S(\sigma)] \end{aligned}$$

be the quotient stack. By [13, Proposition 3.5], ${}^P\mathcal{X}(\Sigma/\sigma)$ is a closed substack of \mathcal{P} .

Proposition 3.2 ([13], Proposition 3.6). *Let $\pi : \mathcal{P} \rightarrow B$ be a toric stack bundle over a smooth variety B with fibre the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ associated to the extended stacky fan Σ , then the inertia stack of \mathcal{P} is*

$$\mathcal{IP} = \coprod_{b \in \text{Box}(\Sigma)} \mathcal{P}_b := \coprod_{b \in \text{Box}(\Sigma)} {}^P\mathcal{X}(\Sigma/\sigma(\bar{b})).$$

The age of \mathcal{P}_b is the same as the age of $\mathcal{X}(\Sigma)_b$.

For the principal $(\mathbb{C}^\times)^{n+m}$ -bundle $P = \oplus_{j=1}^{n+m} L_j^*$ over B , where L_j is the corresponding j -th line bundle. Let

$$(9) \quad U_j = \begin{cases} u_j - c_1(L_j) & 1 \leq j \leq n; \\ 0 & n+1 \leq j \leq n+m. \end{cases}$$

By abuse of notation, we also denote U_j for the corresponding \mathbb{T} -equivariant line bundle over \mathcal{P} .

3.2. Main result. We choose an integral basis $\{p_1, \dots, p_{n-r}\}$ of \mathbb{L}^\vee . The toric stack bundle \mathcal{P} is endowed with $n-r$ tautological line bundles whose first Chern classes we denote by $-P_1, \dots, -P_{n-r}$. They restrict to the corresponding first Chern classes $-p_1, \dots, -p_{n-r}$ on the fiber. Recall that the \mathbb{T} -equivariant Novikov ring of the toric stack $\mathcal{X}(\Sigma)$ is defined as

$$\Lambda_{nov}^\mathbb{T} := S_\mathbb{T}[[\overline{\text{NE}}(\mathcal{X}(\Sigma)) \cap H_2(\mathcal{X}(\Sigma); \mathbb{Z})]],$$

where $S_\mathbb{T}$ is the fraction field of $R_\mathbb{T} := H_\mathbb{T}^*(pt, \mathbb{C})$ and $\overline{\text{NE}}(\mathcal{X}(\Sigma))$ is the Mori cone of $\mathcal{X}(\Sigma)$.

For $\mathcal{D} \in H_2(\mathcal{P})$, let $\mathfrak{D} := \pi_*(\mathcal{D}) \in H_2(B)$ be its projection to the base and let

$$\lambda = (d, k) \in \mathbb{L}^S \otimes \mathbb{Q},$$

under the canonical splitting $\mathbb{L}^S \otimes \mathbb{Q} \cong (\mathbb{L} \otimes \mathbb{Q}) \oplus \mathbb{Q}^m$ be the fiber class, such that $\langle P_i, \mathcal{D} \rangle = \langle p_i, d \rangle$. Hence \mathcal{D} is represented by $Q^\mathfrak{D} q^d$ in the Novikov ring of \mathcal{P} .

Let $J_B(z, \tau) = \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} J_\mathfrak{D}(z, \tau) Q^\mathfrak{D}$ be the decomposition of the J function of B according to the degree of curves.

Definition 3.3. We introduce the hypergeometric modification (The S -extended \mathbb{T} -equivariant I -function of the toric stack bundle \mathcal{P})

$$I_\mathcal{P}^S(z, t, \tau, q, x, Q) := e^{\sum_{i=1}^n U_i t_i / z} \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{b \in \text{Box}(\Sigma)} \sum_{\lambda \in \Lambda E_b^S} J_\mathfrak{D}(z, \tau) Q^\mathfrak{D} \tilde{q}^\lambda e^{\lambda t} \left(\prod_{i=1}^{n+m} \frac{\prod_{\langle a \rangle = \langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq 0} (U_i + az)}{\prod_{\langle a \rangle = \langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (U_i + az)} \right) \mathbf{1}_b$$

where

- (i) for each $\lambda \in \Lambda E_b^S$, we write λ_i for the i th component of λ as an element of \mathbb{Q}^{n+m} . We have $\langle \lambda_i \rangle = \hat{b}_i$ for $1 \leq i \leq n$ and $\langle \lambda_i \rangle = 0$ for $n+1 \leq i \leq n+m$.
- (ii) $U_i := 0$, if $n+1 \leq i \leq n+m$.
- (iii) $\mathbf{1}_b$ is the identity class supported on the twisted sector $\mathcal{X}(\Sigma)_b$ associated to $b \in \text{Box}(\Sigma)$;
- (iv) $t = (t_1, \dots, t_n)$ are variables, and $e^{\lambda t} := \prod_{i=1}^n e^{\langle D_i, d \rangle t_i}$
- (v) for $\lambda = (d, k) \in \Lambda E^S \subset \mathbb{L}^S \otimes \mathbb{Q}$, we have $k \in (\mathbb{Z}_{\geq 0})^m$ and $d \in \overline{\text{NE}}(\mathcal{X}(\Sigma)) \cap H_2(\mathcal{X}(\Sigma); \mathbb{Z})$, we write $\tilde{q}^\lambda = q^d x^k = q^d x_1^{k_1} \cdots x_m^{k_m} \in \Lambda_{nov}^\mathbb{T}[[x]]$, with variables $x = (x_1, \dots, x_m)$.

Definition 3.4 ([8]). A $\Lambda_{nov}^{\mathbb{T}}[[x]]$ -valued point of $\mathcal{L}_{\mathcal{P}}$ is an element of $\mathcal{H}[[x]]$ of the form

$$-1z + \mathbf{t}(z) + \sum_{n=0}^{\infty} \sum_{\substack{d \in \overline{\text{NE}}(\mathcal{X}(\Sigma)) \\ \mathfrak{D} \in \overline{\text{NE}}(B)}} \sum_{\alpha} \frac{Q^{\mathfrak{D}} q^d}{n!} \langle \mathbf{t}(\bar{\psi}), \dots, \mathbf{t}(\bar{\psi}), \frac{\phi_{\alpha}}{-z - \bar{\psi}} \rangle_{0, n+1, \mathcal{D}}^{\mathbb{T}} \phi^{\alpha}$$

for some $\mathbf{t}(z) \in \mathcal{H}_+[[x]]$ with $\mathbf{t}|_{Q=q=x=0} = 0$.

The following is the main result of this paper.

Theorem 3.5. *The hypergeometric modification $I_{\mathcal{P}}^S(z, t, \tau, q, x, Q)$ is a $\Lambda_{nov}^{\mathbb{T}}[[x, t]]$ -valued point of the Lagrangian cone $\mathcal{L}_{\mathcal{P}}$ for the \mathbb{T} -equivariant Gromov-Witten theory of \mathcal{P} .*

The rest of this paper is devoted to a proof of Theorem 3.5.

4. LOCALIZATION METHODS IN TORIC GROMOV-WITTEN THEORY

In this Section we describe a characterization of the Lagrangian cone of a toric stack bundle \mathcal{P} via localization.

4.1. Lagrangian cones for toric stack bundles. Given a toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ associated to an extended stacky fan Σ . The maximal torus \mathbb{T} acts on the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$, hence acts on the toric stack bundle $\mathcal{P} = {}^P\mathcal{X}(\Sigma)$. The fixed points under the torus action correspond to the top dimensional cones in the fan Σ . A top dimensional cone σ gives a fixed point section² $\mathcal{P}_{\sigma} := {}^P\mathcal{X}(\Sigma/\sigma)$ for the toric stack bundle \mathcal{P} . Note that \mathcal{P}_{σ} is an abelian gerbe over the base B : it is a fiber product of root gerbes associated to the line bundles defining \mathcal{P} . We write $N_{\sigma}\mathcal{P}$ for the normal bundle at the \mathbb{T} -fixed section \mathcal{P}_{σ} .

For the rest of this paper, we write \mathcal{H} for Givental's symplectic vector space associated to the toric stack bundle \mathcal{P} . Let σ be a top-dimensional cone, we denote Givental's symplectic vector space associated to the \mathbb{T} -fixed section \mathcal{P}_{σ} by \mathcal{H}_{σ} . Let $\mathcal{H}_{\sigma}^{tw}$ and $\mathcal{L}_{\sigma}^{tw}$ be the symplectic vector space and Lagrangian cone associated to the twisted Gromov-Witten theory of \mathcal{P}_{σ} , where the twist is given by the vector bundle $N_{\sigma}\mathcal{P}$ and the \mathbb{T} -equivariant inverse Euler class $e_{\mathbb{T}}^{-1}$. See [18] for more details on twisted theory.

Let

$$\Sigma_{top} := \{\sigma \in \Sigma : \sigma \text{ is a top-dimensional cone in } \Sigma\} \subset \Sigma$$

be the set of top-dimensional cones in Σ . By the Atiyah-Bott localization theorem, we have an isomorphism of Chen-Ruan orbifold cohomology rings

$$(10) \quad H_{\text{CR}, \mathbb{T}}^*(\mathcal{P}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}} \cong \bigoplus_{\sigma \in \Sigma_{top}} H_{\text{CR}}^*(\mathcal{P}_{\sigma}) \otimes_{\mathbb{C}} S_{\mathbb{T}}.$$

In particular, the identity class $1 \in H_{\text{CR}, \mathbb{T}}^*(\mathcal{P})$ corresponds to $\bigoplus_{\sigma \in \Sigma_{top}} 1_{\sigma}$, where 1_{σ} is the identity element in $H_{\text{CR}, \mathbb{T}}^*(\mathcal{P}_{\sigma})$. Furthermore, we have an isomorphism of vector spaces:

$$(11) \quad \mathcal{H} \cong \bigoplus_{\sigma \in \Sigma_{top}} \mathcal{H}_{\sigma}.$$

²We abuse notation here: \mathcal{P}_{σ} are gerbes over B which may not have sections.

For each $f \in \mathcal{H}$ and $\sigma \in \Sigma_{top}$, let $f_\sigma := f|_{\mathcal{H}_\sigma} \in \mathcal{H}_\sigma$ be the restriction of f to the component \mathcal{H}_σ of \mathcal{H} . Hence f_σ can also be viewed as the restriction of f to the inertia stack \mathcal{IP}_σ . Let $f_{(\sigma,b)} := f_\sigma|_{(\mathcal{P}_\sigma)_b}$ be the restriction of f_σ to the twisted sector $(\mathcal{P}_\sigma)_b$ of \mathcal{IP}_σ corresponding to the box element $b \in \text{Box}(\sigma)$.

4.2. Toric virtual localization. We spell out explicitly the virtual localization applied to \mathcal{P} . Our presentation closely follows the toric case in [15].

The \mathbb{T} -action on \mathcal{P} induces a \mathbb{T} -action on the moduli space $\overline{M}_{0,n+1}(\mathcal{P}, \mathcal{D})$. The \mathbb{T} -fixed strata in the moduli space $\overline{M}_{0,n+1}(\mathcal{P}, \mathcal{D})$ are indexed by decorated trees Γ , where Γ consists of the following data.

- (i) each vertex $v \in \Gamma$ is assigned with a top-dimensional cone $\sigma \in \Sigma_{top}$, and we denote the vertex by $v(\sigma)$.
- (ii) each edge $e \in \Gamma$ is assigned with a codimension-1 cone $\tau_e \in \Sigma$.
- (iii) We denote $V(\Gamma)$ to be the set of vertices of Γ , $E(\Gamma)$ to be the set of edges of Γ . Let

$$F(\Gamma) = \{(e, v) \in E(\Gamma) \times V(\Gamma) | e \text{ is incident to } v\}$$

be the set of flags in Γ .

- (iv) Each edge e is associated with a positive integer d_e by the degree map $d : E(\Gamma) \rightarrow \mathbb{Z}_{>0}$.
- (v) Each flag (e, v) of Γ is labelled with an element $k_{(e,v)} \in G_v$, where G_v is the isotropy group of the \mathbb{T} -fixed section \mathcal{P}_σ .
- (vi) There is a marking map $s : \{1, 2, \dots, n+1\} \rightarrow V(\Gamma)$ that associates each marking with vertices of Γ .
- (vii) An element $k_j \in G_{s(j)}$ is associated with the marking $j \in \{1, 2, \dots, n+1\}$.
- (viii) Some compatibility conditions as in [15, Definition 9.6].

Note that the degree \mathcal{D} is encoded in the tree Γ by $\mathfrak{D} = \pi_* \mathcal{D}$ for each vertex v and d_e for each edge e .

We write $DT_{0,n+1}(\mathcal{P}, \mathcal{D})$ for all decorated trees that contain the above data.

For a vertex v in a decorated graph $\Gamma \in DT_{0,n+1}(\mathcal{P}, \mathcal{D})$, we define:

- $S(v) := \{j \in \{1, 2, \dots, n+1\} : s(j) = v\}$, the set of markings associated to the vertex v .
- $E(v) := \{e \in E(\Gamma) : (e, v) \in F(\Gamma)\}$, the set of edges incident to the vertex v .
- $\text{val}(v) := |E(v)| + |S(v)|$, the valence of the vertex v .

We write \mathcal{M}_Γ for the fixed locus of $\overline{M}_{0,n+1}(\mathcal{P}, \mathcal{D})$ given by Γ , the contribution of the Gromov-Witten invariant $\langle \gamma_1 \bar{\psi}_1^{a_1}, \dots, \gamma_{n+1} \bar{\psi}_{n+1}^{a_{n+1}} \rangle_{0,n+1,\mathcal{D}}$ from \mathcal{M}_Γ is:

$$(12) \quad c_\Gamma \prod_{e \in E(\Gamma)} h(e) \prod_{(e,v) \in F(\Gamma)} h(e, v) \prod_{v \in V(\Gamma)} \left(\prod_{j: s(j)=v} \iota_\sigma^* \gamma_j \right) \\ \times \prod_{v \in V(\Gamma)} \int_{[\overline{\mathcal{M}}_{0, \text{val}(v)}(\mathcal{P}_\sigma, \mathfrak{D})]^w} \frac{h(v) \prod_{j \in S(v)} \bar{\psi}_j^{a_j}}{\prod_{e \in E(v)} (e_{\mathbb{T}}(T_{\eta(e,v)} \mathcal{C}_e) - \bar{\psi}_{(e,v)}/r_{(e,v)})}$$

where:

- $c_\Gamma = \frac{1}{|Aut(\Gamma)|} \prod_{e \in E(\Gamma)} \frac{1}{d_e |G_e|} \prod_{(e,v) \in F(\Gamma)} \frac{|G_v|}{r_{(e,v)}}$.
- G_e is the generic stabilizer of the toric substack bundle \mathcal{P}_{τ_e} .
- $r_{(e,v)} := |\langle k_{(e,v)} \rangle|$ is the order of $k_{(e,v)} \in G_v$.
- $h(e) = \frac{e_{\mathbb{T}}(H^1(\mathcal{C}_e, f_e^* T\mathcal{P})^{mov})}{e_{\mathbb{T}}(H^0(\mathcal{C}_e, f_e^* T\mathcal{P})^{mov})}$
- $h(e, v) = e_{\mathbb{T}}((T_\sigma \mathcal{P})^{k_{(e,v)}})$
- $h(v) = e_{\mathbb{T}}^{-1}((N_\sigma \mathcal{P})_{0, val(v), \mathfrak{D}})$
- $f_e : \mathcal{C}_e \rightarrow \mathcal{P}$ is a map to the toric substack bundle $\mathcal{P}_{\tau_e} = {}^P\chi(\Sigma/\tau_e)$.
- $H^i(\mathcal{C}_e, f_e^* T\mathcal{P})^{mov}$ denotes the moving part of $H^i(\mathcal{C}_e, f_e^* T\mathcal{P})$ with respect to the \mathbb{T} -action.
- $\iota_\sigma : \mathcal{P}_\sigma \hookrightarrow \mathcal{P}$ is the inclusion of the fixed section \mathcal{P}_σ .
- $\eta(e, v) = \mathcal{C}_e \cap \mathcal{C}_v$ is a node of \mathcal{C} on \mathcal{C}_e , where $(e, v) \in F(\Sigma)$.
- $\vec{b}(v) \in (G_v)^{val(v)}$ is given by the decorations $k_j, j \in S(v)$, and $k_{(e,v)}, e \in E(v)$.
- $(N_\sigma \mathcal{P})_{0, val(v), 0}$ is the twisting bundle associated to the vector bundle $N_\sigma \mathcal{P}$ over the \mathbb{T} -fixed section \mathcal{P}_σ , as in [18, Definition 2.5.10].
- $\overline{\mathcal{M}}_{0, val(v)}^{\vec{b}(v)}(\mathcal{P}_\sigma, \mathfrak{D})$ is taken to be a point if $val(v) \leq 2$ and $\mathfrak{D} = 0$. The twisting bundles $(N_\sigma \mathcal{P})_{0, val(v), 0}$ in these unstable cases are defined to be $(T_\sigma \mathcal{P})^{k_{e,v}}$, as in the end of [15, Section 9.3.4].

4.3. Characterization theorem. For $\sigma \in \Sigma_{top}$, let $U_k(\sigma)$ be the character of \mathbb{T} given by the restriction of the line bundle U_k to the \mathbb{T} -fixed locus \mathcal{P}_σ .

We will prove the following characterization result:

Theorem 4.1. *Let $\mathcal{P} = {}^P\mathcal{X}(\Sigma)$ be a smooth toric stack bundle associated to an extended stacky fan $\Sigma = (N, \Sigma, \rho)$ and a $(\mathbb{C}^\times)^{n+m}$ bundle $P \rightarrow B$. Let $x = (x_1, \dots, x_m)$ be formal variables. Suppose f is an element of $\mathcal{H}[[x]]$ satisfies $f|_{Q=q=x=0} = -1z$, then f is a $\Lambda_{nov}^{\mathbb{T}}[[x]]$ -value point of the Lagrangian cone $\mathcal{L}_{\mathcal{P}}$ if and only if it meets the following three conditions:*

- (C1): *For each $\sigma \in \Sigma_{top}$ and $b \in \text{Box}(\sigma)$, the restriction $f_{(\sigma,b)}$ is a power series in Q, q and x with coefficients being elements of $S_{\mathbb{T}}(z)$. As a function in z , $f_{(\sigma,b)}$ has essential singularity at $z = 0$, a finite order pole at $z = \infty$, simple poles at $z = \frac{U_j(\sigma)}{c}$, whenever there exists $\sigma' \in \Sigma$ and $c > 0$ such that $\sigma \nmid \sigma'$, $j \in \sigma \setminus \sigma'$ and $\langle c \rangle = \hat{b}_j$. And $f_{(\sigma,b)}$ is regular elsewhere.*
- (C2): *The residues of $f_{(\sigma,b)}$ at the simple pole $z = \frac{U_j(\sigma)}{c}$ satisfy the following recursion relations:*

$$\text{Res}_{z=\frac{U_j(\sigma)}{c}} f_{(\sigma,b)}(z) dz = -q^{d_{c,\sigma,j}} \text{Rec}(c)_{(\sigma,b)}^{(\sigma',b')} f_{(\sigma',b')}(z) \Big|_{z=\frac{U_j(\sigma)}{c}},$$

where the recursion coefficient $\text{Rec}(c)_{(\sigma,b)}^{(\sigma',b')}$ associated to (σ, σ', b, c) is an element of $S_{\mathbb{T}}$ given by:

$$\text{Rec}(c)_{(\sigma,b)}^{(\sigma',b')} := \frac{1}{c} \left(\prod_{i \in \sigma: b_i=0} U_i(\sigma) \right) \frac{\left(\frac{c}{U_j(\sigma)} \right)^{[c]}}{[c]!} \frac{\left(\frac{c}{U_j(\sigma)} \right)^{[c']}}{[c']!} \prod_{i \in \sigma \cap \sigma'} \frac{\prod_{\langle a \rangle = \hat{b}_i, a < 0} (U_i(\sigma) + U_j(\sigma) \frac{a}{-c})}{\prod_{\langle a \rangle = \hat{b}_i, a < c_i} (U_i(\sigma) + U_j(\sigma) \frac{a}{-c})},$$

- (C3): *The Laurent expansion of the restriction f_σ at $z = 0$ is a $\Lambda_{nov}^{\mathbb{T}}[[x]]$ -valued point of the twisted Lagrangian cone \mathcal{L}_σ^{tw} .*

Proof. We will follow the approach in [8]. Let $\{\phi_\alpha\}$ be a basis for $H_{\text{CR},\mathbb{T}}^*(\mathcal{P}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}$ and $\{\phi^\alpha\}$ be its dual basis with respect to the orbifold Poincaré pairing. Suppose f is a $\Lambda_{\text{nov}}^{\mathbb{T}}[[x]]$ -valued point on the Lagrangian cone $\mathcal{L}_{\mathcal{P}}$. Then f can be written as

$$(13) \quad f = -1z + \mathbf{t}(z) + \sum_{n=0}^{\infty} \sum_{\substack{d \in \overline{\text{NE}}(\mathcal{X}(\Sigma)) \\ \mathfrak{D} \in \overline{\text{NE}}(B)}} \sum_{\alpha} \frac{Q^{\mathfrak{D}} q^d}{n!} \langle \mathbf{t}(\bar{\psi}), \dots, \mathbf{t}(\bar{\psi}), \frac{\phi_\alpha}{-z - \bar{\psi}} \rangle_{0,n+1,\mathcal{D}}^{\mathbb{T}} \phi^\alpha$$

for some $\mathbf{t}(z) \in \mathcal{H}_+[[x]]$ with $\mathbf{t}|_{Q=q=x=0} = 0$. Under the isomorphism (11), we have that f is determined by its restrictions f_σ to \mathcal{H}_σ :

$$f_\sigma = -1_\sigma z + \mathbf{t}_\sigma(z) + \iota_\sigma^* \left(\sum_{n=0}^{\infty} \sum_{\substack{d \in \overline{\text{NE}}(\mathcal{X}(\Sigma)) \\ \mathfrak{D} \in \overline{\text{NE}}(B)}} \sum_{\alpha} \frac{Q^{\mathfrak{D}} q^d}{n!} \langle \mathbf{t}(\bar{\psi}), \dots, \mathbf{t}(\bar{\psi}), \frac{\phi_\alpha}{-z - \bar{\psi}} \rangle_{0,n+1,\mathcal{D}}^{\mathbb{T}} \phi^\alpha \right)$$

where $\iota_\sigma : \mathcal{P}_\sigma \rightarrow \mathcal{P}$ is the inclusion of the \mathbb{T} -fixed section. Furthermore, let $\phi_{\sigma,b}^\alpha$ be the restriction of ϕ^α to $\mathcal{IP}_{\sigma,b}$, we obtain the following sum over graphs via virtual localization in \mathbb{T} -equivariant cohomology:

$$(14) \quad \begin{aligned} f_{(\sigma,b)} &= -\delta_{b,0}z + \mathbf{t}_{(\sigma,b)}(z) + \sum_{n=0}^{\infty} \sum_{\substack{d \in \overline{\text{NE}}(\mathcal{X}(\Sigma)) \\ \mathfrak{D} \in \overline{\text{NE}}(B)}} \sum_{\alpha} \frac{Q^{\mathfrak{D}} q^d}{n!} \langle \frac{\phi_\alpha}{-z - \bar{\psi}}, \mathbf{t}(\bar{\psi}), \dots, \mathbf{t}(\bar{\psi}) \rangle_{0,n+1,\mathcal{D}}^{\mathbb{T}} \phi_{\sigma,b}^\alpha \\ &= -\delta_{b,0}z + \mathbf{t}_{(\sigma,b)}(z) + \sum_{n=0}^{\infty} \sum_{\substack{d \in \overline{\text{NE}}(\mathcal{X}(\Sigma)) \\ \mathfrak{D} \in \overline{\text{NE}}(B)}} \frac{Q^{\mathfrak{D}} q^d}{n!} \sum_{\Gamma \in DT_{0,n+1}(\mathcal{P},\mathcal{D})} C(\Gamma)_{\sigma,b} \end{aligned}$$

where $C(\Gamma)_{\sigma,b}$ is the contribution from the \mathbb{T} -fixed stratum $\mathcal{M}_\Gamma \subset \overline{M}_{0,n+1}(\mathcal{P},\mathcal{D})$ corresponding to the decorated tree Γ .

$$\sum_{\alpha} \langle \frac{\phi_\alpha}{-z - \bar{\psi}}, \mathbf{t}(\bar{\psi}), \dots, \mathbf{t}(\bar{\psi}) \rangle_{0,n+1,\mathcal{D}}^{\mathbb{T}} \phi_{\sigma,b}^\alpha = \sum_{\Gamma \in DT_{0,n+1}(\mathcal{P},\mathcal{D})} C(\Gamma)_{\sigma,b}.$$

In each decorated tree Γ , there is a distinguished vertex v that carries the first marked point. We may assume that $v(\sigma) = v$ and the element k_1 associated with the first marking is \hat{b} , otherwise the contribution of Γ is zero. There are two possibilities:

- (A): The irreducible component carrying the first marked point is a ramified cover of a 1-dimensional orbit which lies in a fiber \mathcal{X} of the toric stack bundle $\mathcal{P} \rightarrow B$. In this case $\text{val}(v) = 2$;
- (B): The irreducible component carrying the first marked point maps to a fixed section \mathcal{P}_σ .

Consider a graph Γ of type (A). Let $e \in E(\Gamma)$ be the only edge incident to v . We denote the subgraph $\Gamma \setminus \{v, e\}$ by Γ' , then Γ' is connected with v through the edge e . Let $v' \in V(\Gamma')$ be the other vertex incident to e and $v(\sigma') = v'$. We assume the first marking of the graph Γ' is associated with the vertex v' . For the fixed locus \mathcal{M}_Γ , We have \mathcal{C}_e being a \mathbb{P}^1 toric orbifold and $\mathcal{C}_e \cong \mathbb{P}_{r_{(e,v)}, r_{(e,v')}}$. The map $f_e : \mathcal{C}_e \rightarrow \mathcal{P}$ satisfies $f_e(0) \in \mathcal{P}_\sigma$ and $f_e(\infty) \in \mathcal{P}_{\sigma'}$. Hence,

$f_e(\mathcal{C}_e)$ is in a fiber of \mathcal{P} , therefore $\mathfrak{D} = 0$, where \mathcal{D} is the degree of f and $\mathfrak{D} = \pi_*(\mathcal{D}) \in H_2(B)$. The contribution $C(\Gamma)_{\sigma,b}$ is nontrivial only if

$$\phi_{\alpha}^{\sigma, \hat{b}} = |N(\sigma)| e_{\mathbb{T}}(N_{\sigma,b}) 1_{\sigma, \hat{b}} \text{ and } \phi_{\sigma,b}^{\alpha} = [\mathcal{IP}_{\sigma,b}],$$

where $[\mathcal{IP}_{\sigma,b}]$ is the fundamental class of $\mathcal{IP}_{\sigma,b}$, $N_{\sigma,b}$ is the normal bundle to $\mathcal{IP}_{\sigma,b}$ in \mathcal{IP}_b and $1_{\sigma, \hat{b}}$ is the fundamental class of $\mathcal{IP}_{\sigma, \hat{b}}$ with $\hat{b} = \text{inv}(b)$. The box element $\hat{b} \in \text{Box}(\sigma)$ is given by the restriction $f_e|_0 : B\mu_{r(e,v)} \rightarrow \mathcal{P}_{\sigma}$. The morphism f_e determines a rational number $c \in \mathbb{Q}$ and a box element $b' \in \text{Box}(\sigma_{v'})$. Since $\bar{\psi}_1 = -r_{(e,v)} e_{\mathbb{T}}(T_{\eta(e,v)} \mathcal{C}_e)$, using (12), we obtain:

$$\begin{aligned} C(\Gamma)_{\sigma,b} &= \frac{c_{\Gamma}}{c_{\Gamma'}} h(e) h(e, v) h(e, v') \\ &\times \int_{[\overline{\mathcal{M}}_{0,2}^{(\hat{b}, b)}(\mathcal{P}_{\sigma}, 0)]^w} \frac{|N(\sigma)| |e_{\mathbb{T}}(N_{\sigma,b})|}{-z + r_{(e,v)} e_{\mathbb{T}}(T_{\eta(e,v)} \mathcal{C}_e)} \frac{1}{(e_{\mathbb{T}}(T_{\eta(e,v)} \mathcal{C}_e) - \bar{\psi}_2/r_{(e,v)})} \cup h(v) \\ &\times \frac{r_{(e,v')}}{|N(\sigma')| |e_{\mathbb{T}}(N_{\sigma',b'})|} C(\Gamma')_{\sigma',b'}|_{z=-r_{(e,v')} (e_{\mathbb{T}}(T_{\eta(e,v')} \mathcal{C}_e))} \end{aligned}$$

Using [15, (9.14)] and the definition of c_{Γ} , $h(e)$, $h(e, v)$, $h(v)$, we write this as:

$$\begin{aligned} C(\Gamma)_{\sigma,b} &= \frac{|G_v|}{d_e |G_e|} h(e) \frac{e_{\mathbb{T}}(N_{\sigma,b})}{(-z + U_j(\sigma)/c)} C(\Gamma')_{\sigma',b'}|_{z=-r_{(e,v')} e_{\mathbb{T}}(T_{\eta(e,v')} \mathcal{C}_e)} \\ &= \frac{\text{Rec}(c)_{(\sigma,b)}^{(\sigma',b')}}{(-z + U_j(\sigma)/c)} C(\Gamma')_{\sigma',b'}|_{z=U_j(\sigma)/c}. \end{aligned}$$

Hence, the contribution to $f_{(\sigma,b)}$ from all graphs Γ of type **(A)** is:

$$(15) \quad \sum_{\sigma': \sigma \dagger \sigma'} \sum_{\substack{c \in \mathbb{Q}: c > 0, \\ \langle c \rangle = \hat{b}_j}} q^{d_{c,\sigma,j}} \frac{\text{Rec}(c)_{(\sigma,b)}^{(\sigma',b')}}{(-z + U_j(\sigma)/c)} [f_{(\sigma',b')}]_{z=U_j(\sigma)/c}.$$

Hence we have proved **(C2)**, as well as **(C1)**.

To prove **(C3)**, we define: $t_{\sigma}(z) := \sum_{b \in \text{Box}(\sigma)} t_{(\sigma,b)}(z) 1_b$, where

$$t_{(\sigma,b)}(z) := \mathbf{t}_{(\sigma,b)}(z) + \sum_{\sigma': \sigma \dagger \sigma'} \sum_{\substack{c \in \mathbb{Q}: c > 0, \\ \langle c \rangle = \hat{b}_j}} q^{d_{c,\sigma,j}} \frac{\text{Rec}(c)_{(\sigma,b)}^{(\sigma',b')}}{(-z + U_j(\sigma)/c)} [f_{(\sigma',b')}]_{z=U_j(\sigma)/c},$$

and $t_{(\sigma,b)}(z)$ is expanded in terms of positive powers of z .

Then, f_{σ} can be written as:

$$(16) \quad \sum_{b \in \text{Box}(\sigma)} f_{(\sigma,b)} 1_b = -1_{\sigma} z + t_{\sigma}(z) + \sum_{n=0}^{\infty} \sum_{\substack{d \in \overline{\text{NE}}(\mathcal{X}(\Sigma)) \\ \mathfrak{D} \in \overline{\text{NE}}(B)}} \sum_{b \in \text{Box}(\sigma)} \sum_{\substack{\Gamma \in DT_{0,n+1}(\mathcal{P}, \mathcal{D}) \\ \Gamma \text{ is of type B}}} \frac{Q^{\mathfrak{D}} q^d}{n!} C(\Gamma)_{\sigma,b}.$$

Then, we consider the contribution given by decorated trees Γ of type **(B)** such that $\text{val}(v) = l$, where v is the distinguished vertex. The element k_1 associated to the first marking is

$\hat{b} \in \text{Box}(\sigma)$. By integrating over all the factors $\overline{\mathcal{M}}_{0, \text{val}(v')}^{\vec{b}'}(\mathcal{P}_{\sigma'})$ except those associated with the distinguished vertex v , we can write these contributions as:

$$\sum_{\alpha} \frac{1}{\text{Aut}(\Gamma_2, \dots, \Gamma_l)} \left(\int_{[\overline{\mathcal{M}}_{0,l}^{\hat{b}, b^2, \dots, b^l}(\mathcal{P}_{\sigma}, \mathfrak{D})]^w} \frac{\phi_{\alpha}^{\sigma, \hat{b}}}{-z - \bar{\psi}} \cup p_2(\mathbf{t}, \bar{\psi}_2) \cup \dots \cup p_l(\mathbf{t}, \bar{\psi}_l) \cup e_{\mathbb{T}}^{-1}((N_{\sigma} \mathcal{P})_{0,l, \mathfrak{D}}) \right) \phi_{\sigma, b}^{\alpha}$$

for some box elements $b^2, \dots, b^l \in \text{Box}(\sigma)$ and some polynomials $p_i(\mathbf{t}, \bar{\psi}_i)$ in t_0, t_1, \dots, Q, q and $\bar{\psi}_i$. The graph Γ is obtained from joining type **(A)** subgraphs $\Gamma_2, \dots, \Gamma_l$ at the vertex v . More precisely, Γ_i , for $2 \leq i \leq l$, is of type **(A)** and satisfies one of the following:

- Γ_i consists of the distinguished vertex v and two markings with the first marking coincides with the first marking of Γ . $\text{val}(v) = 2$.
- Γ_i contains the distinguished vertex v with exactly one marking that coincides with the first marking of Γ and exactly one edge e_i connecting v with the rest of the graph. $\text{val}(v) = 2$.

If Γ_i consists of one vertex with two markings, then $p_i(\mathbf{t}, \bar{\psi}_i) = \mathbf{t}_{(\sigma, b^i)}(\bar{\psi}_i)$. Otherwise,

$$p_i(\mathbf{t}, \bar{\psi}_i) = Q^{\mathfrak{D}} q^{d_i} C(\Gamma_i)_{\sigma, b^i} |_{z=\bar{\psi}_i}$$

where d_i is the degree from the subgraph Γ_i . Summing over the contribution $C(\Gamma)_{\sigma, b}$ over all Γ such that $\text{val}(v) = l$ gives the contribution

$$\sum_{\alpha} \frac{1}{(l-1)!} \left(\int_{[\overline{\mathcal{M}}_{0,l}^{\hat{b}, b^2, \dots, b^l}(\mathcal{P}_{\sigma}, t_{\sigma}^* \mathfrak{D})]^w} \frac{\phi_{\alpha}^{\sigma, \hat{b}}}{-z - \bar{\psi}} \cup t_{\sigma}(\bar{\psi}_2) \cup \dots \cup t_{\sigma}(\bar{\psi}_l) \cup e_{\mathbb{T}}^{-1}((N_{\sigma} \mathcal{P})_{0,l, \mathfrak{D}}) \right) \phi_{\sigma, b}^{\alpha}$$

in (16). Hence, we have

$$f_{\sigma} = -1_{\sigma} z + t_{\sigma}(z) + \sum_{l=1}^{\infty} \sum_{\mathfrak{D} \in \text{NE}(\mathcal{P}_{\sigma})} \sum_{b \in \text{Box}(\sigma)} \sum_{\alpha} \frac{1}{(l-1)!} \langle \frac{\phi_{\alpha}^{\sigma, \hat{b}}}{-z - \bar{\psi}}, t_{\sigma}(\psi), \dots, t_{\sigma}(\psi) \rangle_{0,l, \mathfrak{D}}^{\text{tw}} \phi_{\sigma, b}^{\alpha} \in \mathcal{L}_{\sigma}^{tw}$$

i.e., the Laurent expansion at $z = 0$ of f_{σ} lies in the twisted Lagrangian cone $\mathcal{L}_{\sigma}^{tw}$. Thus we have proved **(C3)**.

To prove the other direction of the theorem, we assume that $f \in \mathcal{H}[[x]]$ with $f|_{Q=q=x=0} = -1z$ satisfies conditions **(C1)**, **(C2)**, and **(C3)**. Then, from conditions **(C1)** and **(C2)**, we obtain that:

$$(17) \quad f_{\sigma} = -1_{\sigma} z + \mathbf{t}_{\sigma} + \sum_{b \in \text{Box}(\sigma)} 1_b \sum_{\sigma': \sigma \dagger \sigma'} \sum_{\substack{c \in \mathbb{Q}: c > 0, \\ \langle c \rangle = \hat{b}_j}} q^{d_{c, \sigma, j}} \frac{RC(c)_{(\sigma, b)}^{(\sigma', b')}}{(-z + U_j(\sigma)/c)} [f_{(\sigma', b')}]_{z=U_j(\sigma)/c} + O(z^{-1})$$

for some $\mathbf{t}_{\sigma} \in \mathcal{H}_{\sigma, +}[[x]]$ satisfying $\mathbf{t}_{\sigma}|_{Q=q=t=0} = 0$. The remainder $O(z^{-1})$ is a formal power series in Q, q and x with coefficients in $z^{-1} S_{\mathbb{T}}[z^{-1}]$. Let F be a $\Lambda_{nov}^{\mathbb{T}}[[x]]$ -valued point on $\mathcal{L}_{\mathcal{P}}$ defined by (13) with $\mathbf{t} = \tau$, where $\tau \in \mathcal{H}_+[[x]]$ is the unique element such that its restriction to \mathcal{IP}_{σ} is \mathbf{t}_{σ} . Then, we know that F and f both satisfy conditions **(C1-C3)**, and they have the same restriction \mathbf{t}_{σ} in \mathcal{IP}_{σ} . Hence, it remains to show that f can be uniquely determined by the set of elements $\{\mathbf{t}_{\sigma}\}_{\sigma \in \Sigma_{top}}$.

To prove the uniqueness, we use induction on the degree with respect to Q, q and x . Choose a Kähler class ω of \mathcal{P} , recall that the degree of the monomial $Q^{\mathfrak{D}} q^d x_1^{k_1} \dots x_m^{k_m}$, can be

defined as $\langle \mathcal{D}, \omega \rangle + \sum_{i=1}^m k_i$. Let κ_0 denote the minimal degree of a non-trivial stable map to \mathcal{P} . Suppose that f is uniquely determined from the collection $\{t_\sigma\}_{\sigma \in \Sigma_{top}}$ up to order κ . By the isomorphism (11), we know that f is uniquely determined by the collection of its restrictions $\{f_\sigma\}$, hence to show f is determined up to order $\kappa + \kappa_0$, we just need to show f_σ is determined up to order $\kappa + \kappa_0$. We know by (17) that f_σ is determined up to order $\kappa + \kappa_0$ except for the remainder $O(z^{-1})$. On the other hand, since the Laurent expansion at $z = 0$ of f_σ lies in \mathcal{L}_σ^{tw} , equation (13) implies that the higher order terms $O(z^{-1})$ of z^{-1} is also uniquely determined up to order $\kappa + \kappa_0$. The proof is completed. \square

5. PROOF OF THE MAIN THEOREM

To prove Theorem 3.5, it suffices to show the S -extended I -function $I_{\mathcal{P}}^S(z, t, \tau, q, x, Q)$ satisfies conditions **(C1)**–**(C3)** in Theorem 4.1. Recall that the definition of $I_{\mathcal{P}}^S(z, t, \tau, q, x, Q)$ is in Definition 3.3. Let I_σ^S and $I_{(\sigma, b)}^S$ denote the restrictions of $I_{\mathcal{P}}^S(z, t, \tau, q, x, Q)$ to the inertia stack \mathcal{IP}_σ and the component $(\mathcal{P}_\sigma)_b$ of the inertia stack \mathcal{IP}_σ respectively.

5.1. Condition (C1): Poles of I -function. By Definition 3.3, we have

(18)

$$I_{(\sigma, b)}^S = e^{\sum_{i=1}^n U_i(\sigma) t_i / z} \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\lambda \in \Lambda E_b^S} J_{\mathfrak{D}}(z, \tau) Q^{\mathfrak{D}} \tilde{q}^\lambda e^{\lambda t} \\ \times \left(\prod_{i \in \sigma} \frac{\prod_{\langle a \rangle = \langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq 0} (U_i(\sigma) + az)}{\prod_{\langle a \rangle = \langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (U_i(\sigma) + az)} \right) \left(\prod_{i \notin \sigma} \frac{\prod_{\langle a \rangle = 0, a \leq 0} (az)}{\prod_{\langle a \rangle = 0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (az)} \right)$$

where we identify the top-dimensional cone σ with the index set of its 1-cones and consider $\sigma \subset \{1, \dots, n\}$ as a subset of $\{1, \dots, n + m\}$. Note that $U_i(\sigma) = 0$ for $i \notin \sigma$. We also need to have $\lambda_i - c_1(L_i) \cdot \mathfrak{D} \geq 0$ for $i \notin \sigma$, otherwise the contribution is zero. Therefore, we can see that $I_{(\sigma, b)}^S$ has essential singularity at $z = 0$ and a finite order pole at $z = \infty$ and simple poles at

$$z = -U_i(\sigma)/a \quad \text{with} \quad 0 < a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}, \langle a \rangle = \langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle = \langle \lambda_i \rangle = \hat{b}_i, i \in \sigma,$$

for $\lambda \in \Lambda E_b^S$ contributing to the sum. To see it satisfies **(C1)** of Theorem 4.1, it remains to invoke the following lemma, which is proved in [8, Section 7.1]:

Lemma 5.1. *Consider a top dimensional cone $\sigma \in \Sigma_{top}$, if $\lambda_{i_0} > 0$ for some $i_0 \in \sigma$, then there exists another top-dimensional cone $\sigma' \in \Sigma_{top}$, such that $\sigma \nmid \sigma'$ and $i_0 \in \sigma \setminus \sigma'$.*

5.2. Condition (C2): Recursion relations. let $\sigma, \sigma' \in \Sigma_{top}$ be top-dimensional cones with $\sigma \nmid \sigma'$. Let $b \in \text{Box}(\sigma)$ and fix a positive rational number c such that $\langle c \rangle = \hat{b}_j$. We study the residue of $I_{(\sigma, b)}^S$ at $z = -U_j(\sigma)/c$. Write

$$\Delta_{\lambda, i, \sigma, \mathfrak{D}}(z) := \frac{\prod_{\langle a \rangle = \langle \lambda_i \rangle, a \leq 0} U_i(\sigma) + az}{\prod_{\langle a \rangle = \langle \lambda_i \rangle, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} U_i(\sigma) + az}$$

for $\lambda \in \Lambda^S$, $1 \leq i \leq n + m$, and $\mathfrak{D} \in \overline{\text{NE}}(B)$. The residue of (18) is given by:

$$(19) \quad e^{\frac{\sum_{i=1}^n U_i(\sigma) t_i}{-U_j(\sigma)/c}} \frac{1}{c} \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda E_b^S \\ \lambda_j \geq c}} J_{\mathfrak{D}}(-U_j(\sigma)/c, \tau) Q^{\mathfrak{D}} \tilde{q}^{\lambda} e^{\lambda t} \frac{\prod_{i:i \neq j} \Delta_{\lambda, i, \sigma, \mathfrak{D}}(-U_j(\sigma)/c)}{\prod_{\substack{0 < a \leq \lambda_j - c_1(L_j) \cdot \mathfrak{D}, \langle a \rangle = \langle \lambda_j \rangle \\ a \neq c}} (U_j(\sigma) - a \frac{U_j(\sigma)}{c})}.$$

Consider the change of variables

$$\lambda = \lambda' + d_{c, \sigma, j}$$

where $\lambda' \in \Lambda_{b'}^S$. We write

$$c_i = D_i \cdot d_{c, \sigma, j}, \text{ for } 1 \leq i \leq n; \quad c_j = c; \quad c_{j'} = c'; \quad c_i = 0, \text{ for } n+1 \leq i \leq n+m.$$

For $1 \leq i \leq n$, consider the representable morphism $f : \mathbb{P}_{r_1, r_2} \rightarrow \mathcal{P}$ given by Proposition 2.4 with $f(0) \in \mathcal{P}_{\sigma}$ and $f(\infty) \in \mathcal{P}_{\sigma'}$, then applying the localization formula, we have

$$c_i = D_i \cdot d_{c, \sigma, j} = \int_{\mathbb{P}_{r_1, r_2}} f^* D_i = \int_{\mathbb{P}_{r_1, r_2}} f^* U_i = \frac{U_i(\sigma)}{U_j(\sigma)/c} + \frac{U_i(\sigma')}{-U_j(\sigma')/c'} = \frac{U_i(\sigma)}{U_j(\sigma)/c} + \frac{U_i(\sigma')}{-U_j(\sigma)/c}$$

Hence we obtain

$$(20) \quad U_i(\sigma) = U_i(\sigma') + \frac{c_i}{c} U_j(\sigma).$$

Hence, by equation (20), we have the following three equations

$$(21) \quad \frac{\sum_{i=1}^n U_i(\sigma) t_i}{-U_j(\sigma)/c} + \lambda t = \frac{\sum_{i=1}^n U_i(\sigma') t_i}{-U_j(\sigma)/c} + \lambda' t;$$

$$(22) \quad \Delta_{\lambda, i, \sigma, \mathfrak{D}} \left(-\frac{U_j(\sigma)}{c} \right) = \Delta_{\lambda', i, \sigma', \mathfrak{D}} \left(-\frac{U_j(\sigma)}{c} \right) \frac{\prod_{a \leq 0, \langle a \rangle = \langle \lambda_i \rangle} (U_i(\sigma) - \frac{a}{c} U_j(\sigma))}{\prod_{a \leq c_i, \langle a \rangle = \langle \lambda_i \rangle} (U_i(\sigma) - \frac{a}{c} U_j(\sigma))}, \quad \text{for } i \neq j;$$

$$(23) \quad \prod_{\substack{0 < a \leq \lambda_j - c_1(L_j) \cdot \mathfrak{D}, \\ \langle a \rangle = \langle \lambda_j \rangle, a \neq c}} \left(U_j(\sigma) - a \frac{U_j(\sigma)}{c} \right) = \prod_{\substack{-c < a \leq \lambda'_j - c_1(L_j) \cdot \mathfrak{D}, \\ \langle a \rangle = \langle \lambda_j \rangle, a \neq 0}} \left(-a \frac{U_j(\sigma)}{c} \right).$$

Applying the above three equations we see that (19), the residue of $I_{(\sigma, b)}^S$ at $z = -\frac{U_j(\sigma)}{c}$, is given by:

$$\begin{aligned} & e^{\frac{\sum_{i=1}^n U_i(\sigma') t_i}{-U_j(\sigma)/c}} \frac{1}{c} \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda' \in \Lambda E_{b'}^S \\ \lambda'_j \geq 0}} J_{\mathfrak{D}}(-U_j(\sigma)/c, \tau) Q^{\mathfrak{D}} \tilde{q}^{\lambda'} q^{d_{c, \sigma, j}} e^{\lambda' t} \frac{\prod_{i:i \neq j} \Delta_{\lambda', i, \sigma', \mathfrak{D}}(-U_j(\sigma)/c)}{\prod_{\substack{0 < a \leq \lambda'_j - c_1(L_j) \cdot \mathfrak{D}, \\ \langle a \rangle = \langle \lambda_j \rangle, a \neq 0}} \left(U_j(\sigma) - a \frac{U_j(\sigma)}{c} \right)} \\ & \times \prod_{i:i \neq j} \frac{\prod_{a \leq 0, \langle a \rangle = \langle \lambda_i \rangle} (U_i(\sigma) - \frac{a}{c} U_j(\sigma))}{\prod_{a \leq c_i, \langle a \rangle = \langle \lambda_i \rangle} (U_i(\sigma) - \frac{a}{c} U_j(\sigma))} \\ & = q^{d_{c, \sigma, j}} \frac{1}{c} \frac{1}{\prod_{\substack{0 < a < c, \\ a \in \mathbb{Z}}} \left(a \frac{U_j(\sigma)}{c} \right)} \prod_{i \in \sigma'} \frac{\prod_{a \leq 0, \langle a \rangle = \langle \lambda_i \rangle} (U_i(\sigma) - \frac{a}{c} U_j(\sigma))}{\prod_{a \leq c_i, \langle a \rangle = \langle \lambda_i \rangle} (U_i(\sigma) - \frac{a}{c} U_j(\sigma))} I_{(\sigma', b')}^S|_{z = -U_j(\sigma)/c}. \end{aligned}$$

By a direct computation, we obtain

$$\frac{1}{c} \frac{1}{\prod_{\substack{0 < a < c, \\ a \in \mathbb{Z}}} \left(a \frac{U_j(\sigma)}{c} \right) \prod_{i \in \sigma'} \frac{\prod_{a \leq 0, \langle a \rangle = \langle \lambda_i \rangle} (U_i(\sigma) - \frac{a}{c} U_j(\sigma))}{\prod_{a \leq c_i, \langle a \rangle = \langle \lambda_i \rangle} (U_i(\sigma) - \frac{a}{c} U_j(\sigma))} = \text{Rec}(c)_{(\sigma, b)}^{(\sigma', b')}.$$

This proves the recursion for the S -Extended I -function.

5.3. Condition (C3): Restriction to fixed points. Consider a top dimensional cone $\sigma \in \Sigma_{\text{top}}$, we need to show that I_σ^S lies on the Lagrangian cone \mathcal{L}_σ^{tw} . We will need to use the decomposition theorem of Gromov-Witten theory of μ -gerbe over the base B as in [3].

By Definition 3.3, we have

(24)

$$I_\sigma^S = e^{\sum_{i=1}^n U_i(\sigma) t_i / z} \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda_\sigma^S \\ \lambda_i - c_1(L_i) \cdot \mathfrak{D} \in \mathbb{Z}_{\geq 0} \text{ if } i \notin \sigma}} J_{\mathfrak{D}}(z, \tau) Q^{\mathfrak{D}} \tilde{q}^\lambda e^{\lambda t} \\ \times \left(\prod_{i \in \sigma} \frac{\prod_{\langle a \rangle = \langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq 0} (U_i(\sigma) + az)}{\prod_{\langle a \rangle = \langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (U_i(\sigma) + az)} \right) \left(\prod_{i \notin \sigma} \frac{\prod_{\langle a \rangle = 0, a \leq 0} (az)}{\prod_{\langle a \rangle = 0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (az)} \right) 1_{v^S(\lambda)}$$

where $1_{v^S(\lambda)} \in H_{\text{CR}}^*(\mathcal{P}_\sigma)$ is the identity class on the twisted sector of \mathcal{P}_σ corresponding to the box element $v^S(\lambda) \in \text{Box}(\sigma)$. By string equation, \mathcal{L}_σ^{tw} is invariant under multiplication by $e^{\sum_{i=1}^n U_i(\sigma) t_i / z}$, hence we can remove this factor from (24).

Let $\pi(\sigma)$ be the quotient map $\mathbf{N} \rightarrow \mathbf{N}(\sigma)$. We have

$$v^S(\lambda) = \sum_{j \in \sigma} [\lambda_j] \rho_j + \sum_{i \notin \sigma, i \leq n} \lambda_i \rho_i + \sum_{i=1}^m \lambda_{n+i} s_i \equiv \sum_{i \notin \sigma} \lambda_i b_\sigma^i \pmod{\mathbf{N}_\sigma},$$

where

$$b_\sigma^i = \begin{cases} \pi(\sigma)(\rho_i), & 1 \leq i \leq n; \\ \pi(\sigma)(s_{i-n}), & n+1 \leq i \leq n+m. \end{cases}$$

We also introduce variables $\{q_i\}_{i \notin \sigma}$ and the change of variables:

$$Q^{\mathfrak{D}} \tilde{q}^\lambda e^{\lambda t} = \left(\prod_{i \notin \sigma} q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \right) Q^{\mathfrak{D}}$$

It remains to show that

$$(25) \quad \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda_\sigma^S \\ \lambda_i - c_1(L_i) \cdot \mathfrak{D} \in \mathbb{Z}_{\geq 0} \text{ if } i \notin \sigma}} Q^{\mathfrak{D}} J_{\mathfrak{D}}(z, \tau) \left(\prod_{i \in \sigma} \frac{\prod_{\langle a \rangle = \langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq 0} (U_i(\sigma) + az)}{\prod_{\langle a \rangle = \langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (U_i(\sigma) + az)} \right) \\ \times \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \prod_{\langle a \rangle = 0, a \leq 0} (az)}{\prod_{\langle a \rangle = 0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (az)} \right) 1_{\sum_{i \notin \sigma} \lambda_i b_\sigma^i}$$

is a $S_{\mathbb{T}}[[q]]$ -valued point on the twisted Lagrangian cone \mathcal{L}_σ^{tw} of \mathcal{P}_σ .

By construction, \mathcal{P}_σ is a product of root gerbes over the base B .

Lemma 5.2. *For each top dimensional cone σ , the series*

$$(26) \quad \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{b \in \text{Box}(\sigma)} Q^{\mathfrak{D}} J_{\mathfrak{D}}(z, \tau) 1_b$$

lies on the Lagrangian cone \mathcal{L} of the untwisted theory of \mathcal{P}_{σ}

Proof. By the definition of J-function, we have

$$\begin{aligned} \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{b \in \text{Box}(\sigma)} Q^{\mathfrak{D}} J_{\mathfrak{D}}(z, \tau) 1_b &= \sum_{b \in \text{Box}(\sigma)} (z + \tau + \sum_{n, \mathfrak{D}} \sum_{\alpha} \frac{Q^{\mathfrak{D}}}{n!} \langle \frac{\phi_{\alpha}}{z - \bar{\psi}}, \tau, \dots, \tau \rangle_{0, n+1, \mathfrak{D}}^B \phi^{\alpha}) 1_b \\ &= z \sum_{b \in \text{Box}(\sigma)} \sum_{n, \mathfrak{D}} \sum_{\alpha} \frac{Q^{\mathfrak{D}}}{n!} \langle \frac{\phi_{\alpha}}{z - \bar{\psi}}, 1, \tau, \dots, \tau \rangle_{0, n+2, \mathfrak{D}}^B \phi^{\alpha} \otimes 1_b, \end{aligned}$$

where the second equation is from string equation. By [13, Proposition 5.2], the Chen-Ruan cohomology ring of the toric stack bundle \mathcal{P}_{σ} is given by

$$(27) \quad H_{CR}^*(\mathcal{P}_{\sigma}; \mathbb{Q}) \cong H^*(B; \mathbb{Q}) \otimes H_{CR}^*(BG_{\sigma}; \mathbb{Q})$$

where G_{σ} is the isotropy group at generic points of \mathcal{P}_{σ} . Recall that τ takes value in $H^*(B, \mathbb{Q}) \cong H^*(\mathcal{P}_{\sigma}, \mathbb{Q}) \subset H_{CR}^*(\mathcal{P}_{\sigma}, \mathbb{Q})$. By [2, Theorem 7.1],

$$\langle \frac{\phi_{\alpha}}{z - \bar{\psi}}, 1, \tau, \dots, \tau \rangle_{0, n+2, \mathfrak{D}}^B = |G_{\sigma}| \langle \frac{\phi_{\alpha} \otimes 1_{\hat{b}}}{z - \bar{\psi}}, 1_{\tilde{b}}, \tau, \dots, \tau \rangle_{0, n+2, \mathfrak{D}}^{\mathcal{P}_{\sigma}}$$

where $\hat{b} = \text{inv}(b)$ is the involution of b and \tilde{b} is uniquely determined by the \mathfrak{D} -admissible condition defined in [3, Definition 3.3, Remark 3.7] and [2, Definition 4.5]. Note that \tilde{b} depends on b and \mathfrak{D} . Moreover, by [3, Theorem 4.4] and [2, Theorem 7.1], for each $\tilde{b} \in \text{Box}(\sigma) \setminus \hat{b}$, the invariant $\langle \frac{\phi_{\alpha} \otimes 1_{\hat{b}}}{z - \bar{\psi}}, 1_{\tilde{b}}, \tau, \dots, \tau \rangle_{0, n+2, \mathfrak{D}}^{\mathcal{P}_{\sigma}}$ vanishes. Therefore

$$\langle \frac{\phi_{\alpha} \otimes 1_{\hat{b}}}{z - \bar{\psi}}, 1_{\tilde{b}}, \tau, \dots, \tau \rangle_{0, n+2, \mathfrak{D}}^{\mathcal{P}_{\sigma}} = \langle \frac{\phi_{\alpha} \otimes 1_{\hat{b}}}{z - \bar{\psi}}, \sum_{\underline{b} \in \text{Box}(\sigma)} 1_{\underline{b}}, \tau, \dots, \tau \rangle_{0, n+2, \mathfrak{D}}^{\mathcal{P}_{\sigma}},$$

for every b and \mathfrak{D} . Notice that $\sum_{\underline{b} \in \text{Box}(\sigma)} 1_{\underline{b}}$ does not depend on b or \mathfrak{D} . Then, (26) can be written as

$$z |G_{\sigma}| \sum_{b \in \text{Box}(\sigma)} \sum_{n, \mathfrak{D}} \sum_{\alpha} \frac{Q^{\mathfrak{D}}}{n!} \langle \frac{\phi_{\alpha} \otimes 1_{\hat{b}}}{z - \bar{\psi}}, \sum_{\underline{b} \in \text{Box}(\sigma)} 1_{\underline{b}}, \tau, \dots, \tau \rangle_{0, n+2, \mathfrak{D}}^{\mathcal{P}_{\sigma}} \phi^{\alpha} \otimes 1_b$$

By (27), $\{\phi^{\alpha} \otimes 1_b\}_{b \in \text{Box}_{\sigma, \alpha}}$ forms a basis of $H_{CR}^*(\mathcal{P}_{\sigma}; \mathbb{Q})$, so we have

$$\begin{aligned} &z |G_{\sigma}| \sum_{b \in \text{Box}(\sigma)} \sum_{n, \mathfrak{D}} \sum_{\alpha} \frac{Q^{\mathfrak{D}}}{n!} \langle \frac{\phi_{\alpha} \otimes 1_{\hat{b}}}{z - \bar{\psi}}, \sum_{\underline{b} \in \text{Box}(\sigma)} 1_{\underline{b}}, \tau, \dots, \tau \rangle_{0, n+2, \mathfrak{D}}^{\mathcal{P}_{\sigma}} \phi^{\alpha} \otimes 1_b \\ &= z |G_{\sigma}| S_{\mathcal{P}_{\sigma}}(\tau, z) \left(\sum_{\underline{b} \in \text{Box}(\sigma)} 1_{\underline{b}} \right), \end{aligned}$$

by the definition of the S -operator (1). Hence, we conclude that (26) lies on the Lagrangian cone \mathcal{L} of the untwisted theory of \mathcal{P}_{σ} . \square

Notice that we have the following identity

$$\sum_{\substack{\lambda \in \Lambda_\sigma^S \\ \lambda_i - c_1(L_i) \cdot \mathfrak{D} \in \mathbb{Z}_{\geq 0} \text{ if } i \notin \sigma}} \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \prod_{\langle a \rangle = 0, a \leq 0}(az)}{\prod_{\langle a \rangle = 0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}}(az)} \right) 1_{\sum_{i \notin \sigma} \lambda_i b_\sigma^i} = \sum_{b \in \text{Box}(\sigma)} \exp\left(\sum_{i \notin \sigma} q_i/z\right) 1_b.$$

Hence, we have

$$\begin{aligned} (28) \quad & \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda_\sigma^S \\ \lambda_i - c_1(L_i) \cdot \mathfrak{D} \in \mathbb{Z}_{\geq 0} \text{ if } i \notin \sigma}} Q^\mathfrak{D} J_\mathfrak{D}(z, \tau) \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \prod_{\langle a \rangle = 0, a \leq 0}(az)}{\prod_{\langle a \rangle = 0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}}(az)} \right) 1_{\sum_{i \notin \sigma} \lambda_i b_\sigma^i} \\ &= \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{b \in \text{Box}(\sigma)} Q^\mathfrak{D} \exp\left(\sum_{i \notin \sigma} q_i/z\right) J_\mathfrak{D}(z, \tau) 1_b. \end{aligned}$$

lies on the untwisted Lagrangian cone \mathcal{L} of \mathcal{P}_σ .

We will need to use Tseng's orbifold quantum Riemann-Roch theorem in [18] to prove (25) lies in the twisted Lagrangian cone \mathcal{L}_σ^{tw} . We recall some notations here:

Let V be the direct sum of d vector bundles $V^{(j)}$, for $1 \leq j \leq d$, and consider a universal multiplicative characteristic class:

$$\mathbf{c}(V) = \prod_{j=1}^d \exp\left(\sum_{k=0}^{\infty} s_k^{(j)} ch_k(V^{(j)})\right)$$

where $s_0^{(j)}, s_1^{(j)}, s_2^{(j)}, \dots$ are formal indeterminates. We consider the special case where $V = N_\sigma \mathcal{P}$, which is the direct sum of line bundles $U_j(\sigma)$, for $j \in \sigma$, over \mathcal{P}_σ . For $j \in \sigma$, we set

$$s_k^{(j)} = \begin{cases} -\log U_j(\sigma), & k = 0 \\ (-1)^k (k-1)! U_j(\sigma)^{-k}, & k \geq 1 \end{cases}$$

Then, we obtain the $(N_\sigma \mathcal{P}, e_{\mathbb{T}-1})$ -twisted Gromov-Witten theory of \mathcal{P}_σ . Recall that \mathcal{L}^{tw} is the Lagrangian cone of the $(N_\sigma \mathcal{P}, e_{\mathbb{T}-1})$ -twisted Gromov-Witten theory of \mathcal{P}_σ . By direct computation, we obtain the following equation:

$$(29) \quad \exp(s^{(j)}(x)) = (U_j(\sigma) + x)^{-1},$$

where $s^{(j)}(x) := \left(\sum_{k=0}^{\infty} s_k^{(j)} \frac{x^k}{k!}\right)$.

As in [9], we introduce the function:

$$G_y^{(j)}(x, z) := \sum_{l, m \geq 0} s_{l+m-1}^{(j)} \frac{B_m(y)}{m!} \frac{x^l}{l!} z^{m-1} \in \mathbb{C}[y, x, z, z^{-1}][[s_0^{(j)}, s_1^{(j)}, s_2^{(j)}, \dots]],$$

By [9], the function $G_y^{(j)}(x, z)$ satisfies the following two relations:

$$(30) \quad G_y^{(j)}(x, z) = G_0^{(j)}(x + yz, z);$$

$$(31) \quad G_0^{(j)}(x + z, z) = G_0^{(j)}(x, z) + s^{(j)}(x).$$

Let $\theta_j = \left(\sum_{i \notin \sigma} c_{ij} q_i (\partial / \partial q_i) \right)$, where rational numbers c_{ij} for $i \notin \sigma$ and $j \in \sigma$ are defined by

$$\bar{\rho}_i = \sum_{j \in \sigma} c_{ij} \bar{\rho}_j, \text{ for } 1 \leq i \leq n; \quad \bar{s}_i = \sum_{j \in \sigma} c_{ij} \bar{\rho}_j, \text{ for } 1 \leq i \leq m.$$

Hence, rational numbers c_{ij} satisfy the following equation:

$$\lambda_j = - \sum_{i \notin \sigma} c_{ij} \lambda_i, \text{ for } \lambda \in \Lambda_\sigma^S \text{ and } j \in \sigma.$$

We apply the differential operator $\exp(-\sum_{j \in \sigma} G_0^{(j)}(z\theta_j, z))$ to (28), then we have:

$$\begin{aligned} \mathbf{L} &:= \exp \left(- \sum_{j \in \sigma} G_0^{(j)}(z\theta_j, z) \right) \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda_\sigma^S \\ \lambda_i - c_1(L_i) \cdot \mathfrak{D} \in \mathbb{Z}_{\geq 0} \text{ if } i \notin \sigma}} Q^\mathfrak{D} J_\mathfrak{D}(z, \tau) \\ &\quad \times \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \prod_{\langle a \rangle=0, a \leq 0} (az)}{\prod_{\langle a \rangle=0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (az)} \right) 1_{\sum_{i \notin \sigma} \lambda_i b_\sigma^i} \\ &= \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda_\sigma^S \\ \lambda_i - c_1(L_i) \cdot \mathfrak{D} \in \mathbb{Z}_{\geq 0} \text{ if } i \notin \sigma}} Q^\mathfrak{D} J_\mathfrak{D}(z, \tau) \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \prod_{\langle a \rangle=0, a \leq 0} (az)}{\prod_{\langle a \rangle=0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (az)} \right) \\ &\quad \times \exp \left(- \sum_{j \in \sigma} G_0^{(j)}(-z(\lambda_j - c_1(L_j) \cdot \mathfrak{D}), z) \right) 1_{\sum_{i \notin \sigma} \lambda_i b_\sigma^i} \end{aligned}$$

lies on the untwisted Lagrangian cone \mathcal{L} of \mathcal{P}_σ . On the other hand, Tseng's orbifold quantum Riemann-Roch operator for $\oplus_{j \in \sigma} U_j(\sigma)$ is of the form:

$$\Delta_{tw} := \bigoplus_{b \in \text{Box}(\sigma)} \exp \left(\sum_{j \in \sigma} G_{b_j}^{(j)}(0, z) \right)$$

This operator Δ_{tw} maps the untwisted Lagrangian cone \mathcal{L} to the twisted Lagrangian cone \mathcal{L}^{tw} . Therefore

$$\begin{aligned}
\Delta_{tw} \mathbf{L} &= \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda_{\sigma}^S \\ \lambda_i \in \mathbb{Z} \text{ if } i \notin \sigma}} Q^{\mathfrak{D}} J_{\mathfrak{D}}(z, \tau) \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \prod_{\langle a \rangle=0, a \leq 0} (az)}{\prod_{\langle a \rangle=0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (az)} \right) \\
&\quad \times \exp \left(- \sum_{j \in \sigma} \left(G_{b_j}^{(j)}(0, z) - G_0^{(j)}(-z(\lambda_j - c_1(L_j) \cdot \mathfrak{D}), z) \right) \right) 1_{\sum_{i \notin \sigma} \lambda_i b_{\sigma}^i} \\
&= \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda_{\sigma}^S \\ \lambda_i \in \mathbb{Z} \text{ if } i \notin \sigma}} Q^{\mathfrak{D}} J_{\mathfrak{D}}(z, \tau) \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \prod_{\langle a \rangle=0, a \leq 0} (az)}{\prod_{\langle a \rangle=0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (az)} \right) \\
&\quad \times \exp \left(- \sum_{j \in \sigma} \left(G_0^{(j)}(\langle -\lambda_j \rangle z, z) - G_0^{(j)}(-z(\lambda_j - c_1(L_j) \cdot \mathfrak{D}), z) \right) \right) 1_{\sum_{i \notin \sigma} \lambda_i b_{\sigma}^i} \\
&= \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda_{\sigma}^S \\ \lambda_i \in \mathbb{Z} \text{ if } i \notin \sigma}} Q^{\mathfrak{D}} J_{\mathfrak{D}}(z, \tau) \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \prod_{\langle a \rangle=0, a \leq 0} (az)}{\prod_{\langle a \rangle=0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (az)} \right) \\
&\quad \times \left(\prod_{i \in \sigma} \frac{\prod_{\langle a \rangle=\langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq 0} \exp(-s^{(j)}(-az))}{\prod_{\langle a \rangle=\langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} \exp(-s^j(-az))} \right) 1_{\sum_{i \notin \sigma} \lambda_i b_{\sigma}^i} \\
&= \sum_{\mathfrak{D} \in \overline{\text{NE}}(B)} \sum_{\substack{\lambda \in \Lambda_{\sigma}^S \\ \lambda_i \in \mathbb{Z} \text{ if } i \notin \sigma}} Q^{\mathfrak{D}} J_{\mathfrak{D}}(z, \tau) \left(\prod_{i \notin \sigma} \frac{q_i^{\lambda_i - c_1(L_i) \cdot \mathfrak{D}} \prod_{\langle a \rangle=0, a \leq 0} (az)}{\prod_{\langle a \rangle=0, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (az)} \right) \\
&\quad \times \left(\prod_{i \in \sigma} \frac{\prod_{\langle a \rangle=\langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq 0} (U_i(\sigma) + az)}{\prod_{\langle a \rangle=\langle \lambda_i - c_1(L_i) \cdot \mathfrak{D} \rangle, a \leq \lambda_i - c_1(L_i) \cdot \mathfrak{D}} (U_i(\sigma) + az)} \right) 1_{\sum_{i \notin \sigma} \lambda_i b_{\sigma}^i} = I_{\sigma}^S e^{-\sum_{i=1}^n U_i(\sigma) t_i / z}
\end{aligned}$$

lies on \mathcal{L}^{tw} , where the second equation follows from (30), the third equation follows from (31) and fourth equation follows from (29). This completes the proof of Theorem 3.5.

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